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The dispersion and attenuation of helicon waves in a uniform cylindrical plasma

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A systematic account is given of the derivation of the dispersion relation for helicon waves in a uniform cylindrical plasma bounded by a vacuum. By retaining finite resistivity in the equations, boundary conditions present no difficulties, since the wave magnetic field is continuous through the plasma-vacuum interface. Two unexpected results are found. First, the wave attenuation remains finite in the limit of vanishing resistivity. This is due to the energy dissipated at the interface by the surface currents required to match the plasma wave field to the vacuum wave field. Zero wave attenuation for zero resistivity is recovered if electron inertia is included. Secondly, it is found that waves with azimuthal numbers m of opposite sign propagate differently, but the sense of polarization at the axis of the cylinder is independent of the sign of m.

The argument of the dispersion function is complex and numerical results were obtained using a computer. The method of programming is described, and results are given applicable to propagation in metals at low temperatures, or in a typical gas discharge plasma for the $m = 0$ and $m = \pm 1$ modes.

An example of the amplitude of the wave fields as a function of radius is given for the axisymmetric mode, and of amplitude and phase for the $m = \pm 1$ modes.

1. Introduction

The name helicon, proposed by Aigrain (1960), will be used to describe low frequency electro-magnetic waves which propagate in a highly conducting medium such as a metal at low temperatures, or in a gas discharge plasma, in the presence of a strong applied magnetic field. In the presence of the wave field the initially straight lines of force of the applied field become helical. The distinguishing feature of such waves is the minor role played by charged particle inertia; the energy density of the wave field is almost entirely that associated with the magnetic field of the wave.

In an ionized gas, waves with this property are found in the frequency range between the electron and ion cyclotron frequencies, and are often referred to as high-frequency compressional Alfvén waves, or as low-frequency whistlers, the latter name being derived from the falling pitch in the audio-frequency range which is observed in ionospheric studies (Budden 1961, p. 266). Barkhausen (1919) first drew attention to atmospheric whistlers. Eckersley (1935) and

Storey (1953) extensively investigated this phenomenon. The dispersion relation for plane waves is given by the Appleton–Hartree formula (Ratcliffe 1959). More recently helicons have been observed in metals at low temperatures; a necessary condition for propagation being that the electron cyclotron frequency be much greater than the collision frequency (Bowers, Legendy & Rose 1961). Since the latter date, a number of papers on helicon waves have appeared, but the problem of boundary conditions has been avoided by assuming the dispersion relation appropriate to an infinite medium. In the experiments of Chambers & Jones (1962) it would seem difficult to justify their neglect of vacuum fields to the accuracy implied by their results, but the surface effects described below are not present in the experimental configuration used by these authors, namely where the medium is effectively infinite in extent perpendicular to the applied magnetic field. These remarks apply equally to the experiments of Rose, Taylor & Bowers (1962).

In all the low-frequency experiments involving metals the ions are immobile, while in a plasma the inequality $\Omega_e \tau \ll \omega/\Omega_i$ (see appendix A) is usually satisfied and strong surface currents flow when $\Omega_e \tau \gg 1$. (Ω_e and Ω_i are the electron and ion gyro-frequencies respectively, τ is the collision time for electrons, and ω is wave frequency.) In the theory given by Aigrain (1960), Bowers *et al.* (1961) and Cotti, Wyder & Quattropani (1962), the results are correct only in the limit of plane waves when the surface currents, which they ignore, become small. The boundary conditions of Bernstein & Trehan (1960) and Stix (1962, p. 82) are for a plasma and $\Omega_e \tau = \infty$. In this case (see appendix A) displacement of the plasma boundary greatly reduces the surface currents. However, their theory is inapplicable when $\Omega_e \tau \ll \omega/\Omega_i$ where surface currents are important. Propagation of magneto-ionic waves has been studied in a cylindrical plasma by Formato & Gilardini (1962). Cylindrical waves for which $\Omega_e \tau \gg \omega/\Omega_i$ have been studied by Woods (1962, 1964).

It is the object of this paper to obtain the dispersion relation for helicon waves propagating in a cylindrical medium bounded by a vacuum, in the regime where $\Omega_e \tau$ is large compared to unity but small enough for ion motion to be negligible. To solve the problem it is necessary to retain finite resistivity in the equations. The boundary conditions then require continuity of all components of the wave magnetic field.

2. The model and the assumptions

Fluid equations may be obtained from Boltzmann's equation (Spitzer 1962, p. 23), and the modified Ohm's law is

$$\mathscr{E} + \mathbf{v} \times \mathbf{B}_t + \frac{1}{Ne}(\nabla P_e - \mathbf{J} \times \mathbf{B}_t) - \eta \mathbf{J} - \frac{m_e}{Ne^2}\frac{\partial \mathbf{J}}{\partial t} = 0, \tag{1}$$

where only the last term in linearized, and where $\mathbf{B}_t$ is the total magnetic field, $\mathbf{J}$ is the current density, $\mathscr{E}$ is the electric field, N is the electron number density, m_e is the electron mass, P_e is the electron pressure, η is the electrical resistivity, and $\mathbf{v}$ is the plasma mass velocity. The units used are rationalized M.K.S. A cool plasma will be considered for which electron pressure may be neglected. Ion motion will be omitted so that the $\mathbf{v} \times \mathbf{B}$ term in (1) will be dropped. The range of conditions for which this assumption is justified, assuming a uniform plasma, is discussed in Appendix A.

Perturbation about a zero-order state in which the current and electric field vanish gives the linearized form of (1),

$$\mathbf{E} - \frac{1}{Ne}\mathbf{j} \times \mathbf{B} - \eta\mathbf{j} - \frac{m_e}{Ne^2}\frac{\partial \mathbf{j}}{\partial t} = 0, \tag{2}$$

where $\mathbf{E}$ is the perturbed electric field and the other perturbation quantities are denoted by the appropriate small letters. $\mathbf{B}$ is the applied magnetic field. Equation (2) is to be solved in conjunction with Maxwell's equations which, with neglect of displacement current (see appendix A), give for the perturbation quantities

$$\operatorname{curl} \mathbf{b} = \mu_0 \mathbf{j}, \tag{3}$$

$$\operatorname{curl} \mathbf{E} = -\partial \mathbf{b}/\partial t. \tag{4}$$

In practice a plasma may be magnetically confined and the consequent steady surface currents may affect the wave propagation. Appendix B discusses the conditions for neglecting this effect.

3. Boundary conditions

The general solution of the fourth-order equation ((14) and (15)) is the sum of four Bessel functions each multiplied by an arbitrary constant. The general solution of the second-order equation (22) describing the vacuum magnetic field is the sum of two Bessel functions each multiplied by an arbitrary constant, making a total of six arbitrary constants. In general, six boundary conditions are necessary to eliminate these arbitrary constants to find the dispersion relation. The conditions that the magnetic field of the wave be finite at the origin and vanish at infinity require two arbitrary constants to be put equal to zero in the solution for the plasma fields, and one arbitrary constant to be put equal to zero in the solution for the vacuum field. Thus three additional boundary conditions are required to eliminate the three remaining constants.

Integration of equation (3) and $\nabla . \mathbf{b} = 0$ across the boundary leads to the result that the jump $[\mathbf{b}]$ in the magnetic field in crossing the boundary is given by

$$[\mathbf{n} . \mathbf{b}] = 0, \quad [\mathbf{n} \times \mathbf{b}] = \mu_0 \mathbf{j}^*,$$

where $\mathbf{n}$ is the unit vector normal to the boundary and $\mathbf{j}^*$ is the surface current density. With finite electron mass or finite resistivity the current density $\mathbf{j}$ is finite and so surface currents of infinite density cannot occur. The boundary conditions are therefore

$$[b_r] = 0, \quad [b_\theta] = 0, \quad [b_z] = 0. \tag{5}$$

The components of the wave field b_r, b_θ and b_z are therefore continuous across the boundary at $r = a$. By matching the vacuum field and the plasma fields at the boundary the three remaining constants may be eliminated and the dispersion relation found.

4. The dispersion relation

The equations to be solved in cylindrical geometry are

$$\mathbf{E}-\frac{\mathbf{j}\times\mathbf{B}}{Ne}-\eta\mathbf{j}-\frac{m_e}{Ne^2}\frac{\partial\mathbf{j}}{\partial t}=0, \tag{6}$$

$$\nabla\times\mathbf{b}=\mu_0\mathbf{j}, \tag{7}$$

$$\nabla\times\mathbf{E}=-\partial\mathbf{b}/\partial t. \tag{8}$$

Solutions of the form $\mathbf{b}=\hat{b}(r)\exp[i(m\theta+kz-\omega t)]$ are considered.

Taking the curl of equation (6) and using (7) and (8) yields

$$(\omega+i\nu)\nabla\times(\nabla\times\mathbf{b})-k\Omega_e\nabla\times\mathbf{b}+(\omega\omega_p^2/c^2)\mathbf{b}=0, \tag{9}$$

where c is the velocity of light, $\nu=Ne^2\eta/m_e$ is the electron collision frequency, and $\omega_p=(Ne^2/\epsilon_0 m_e)^{\frac{1}{2}}$ is the plasma frequency. It is convenient to define a collision interval τ such that

$$\Omega_e\tau=\Omega_e/\nu=B/Ne\eta.$$

Equation (9) may be factorized and written

$$(\mathrm{curl}-\beta_1)(\mathrm{curl}-\beta_2)\mathbf{b}=0, \tag{10}$$

where β_1 and β_2 are the two roots of the quadratic

$$(\omega+i\nu)\beta^2-\Omega_e k\beta+\omega\omega_p^2/c^2=0. \tag{11}$$

The general solution of (10) is therefore the sum of the solutions of

$$\mathrm{curl}\,\mathbf{b}=\beta_1\mathbf{b} \tag{12}$$

and

$$\mathrm{curl}\,\mathbf{b}=\beta_2\mathbf{b}. \tag{13}$$

Since $\nabla.\mathbf{b}=0$, equations (12) and (13) may be written

$$\nabla^2\mathbf{b}=-\beta_1^2\mathbf{b}, \tag{14}$$

and

$$\nabla^2\mathbf{b}=-\beta_2^2\mathbf{b}, \tag{15}$$

where

$$\beta_{1,2}=\frac{k}{2(\omega+i\nu)}\left\{1\mp\sqrt{\left[1-4\frac{(\omega+i\nu)}{k^2\Omega_e^2}\left(\frac{\omega\omega_p^2}{c^2}\right)\right]}\right\}. \tag{16}$$

The negative sign is to be taken with β_1 and the positive sign with β_2.

The solution of (10) for the z component, $\hat{b}_z(r)$, which is finite at the origin is

$$\hat{b}_z(r)=A_1 J_m(\gamma_1 r)+A_2 J_m(\gamma_2 r), \tag{17}$$

where A_1 and A_2 are amplitude constants, $\gamma_1^2=\beta_1^2-k^2$, $\gamma_2^2=\beta_2^2-k^2$, and J_m is the Bessel function of the first kind of order m.

Equation (14) may be used to obtain the components $\hat{b}_r$ and $\hat{b}_\theta$ in terms of $\hat{b}_z$. These are

$$\begin{aligned}\hat{b}_r=&(iA_1/\gamma_1^2)[m\beta_1 J_m(\gamma_1 r)/r+k\gamma_1 J_m'(\gamma_1 r)]\\&+(iA_2/\gamma_2^2)[m\beta_2 J_m(\gamma_2 r)/r+k\gamma_2 J_m'(\gamma_2 r)],\end{aligned} \tag{18}$$

$$\hat{b}_\theta = -(A_1/\gamma_1^2)\,[mk\,J_m(\gamma_1 r)/r + \beta_1\gamma_1 J_m'(\gamma_1 r)] \\ -(A_2/\gamma_2^2)\,[mk\,J_m(\gamma_2 r)/r + \beta_2\gamma_2 J_m'(\gamma_2 r)]. \tag{19}$$

Equations (18) and (19) will be represented by

$$\hat{b}_r = iA_1 f_r + iA_2 g_r, \tag{20}$$

$$b_\theta = A_1 f_\theta + A_2 g_\theta. \tag{21}$$

In the region exterior to the medium, displacement currents are neglected and there are no conduction currents so that

$$\operatorname{curl}\mathbf{b} = 0.$$

Therefore $\mathbf{b} = \nabla\phi$ where ϕ is a scalar.

Since

$$\nabla.\mathbf{b} = 0,$$

$$\nabla^2\phi = 0. \tag{22}$$

The vacuum fields are therefore given by

$$\hat{b}_r = A_3 k K_m'(kr), \tag{23}$$

$$\hat{b}_\theta = (im/r)\,A_3 K_m(kr), \tag{24}$$

$$\hat{b}_z = ikA_3 K_m(kr), \tag{25}$$

where $K_m(kr)$ is the modified Bessel function of the second kind, such that

$$K_m(kr) \to 0 \quad \text{as} \quad kr \to \infty.$$

By applying the boundary conditions for continuity of b_r, b_θ, b_z (equation (5)) at $r = a$, the arbitrary constants A_1, A_2 and A_3 may be eliminated to give

$$\begin{vmatrix} f_r & g_r & -akK_m(ak) \\ f_\theta & g_\theta & mK_m(ak) \\ J_m(\gamma_1 a) & J_m(\gamma_2 a) & akK_m(ak) \end{vmatrix} = 0. \tag{26}$$

The determinantal equation (26) is the required dispersion relation.

5. Dispersion and attenuation of the axisymmetric wave ($m = 0$)

The attenuation of helicon waves in the limit $\Omega_e\tau \to \infty$ shows an anomalous behaviour which we illustrate for simplicity with the $m = 0$ wave. For $m = 0$ the f and g functions defined by equations (20), (21) take the simple form

$$\left.\begin{aligned} f_r &= -(k/\gamma_1)\,J_1(\gamma_1 a), & g_r &= -(k/\gamma_2)\,J_1(\gamma_2 a), \\ f_\theta &= (\beta_1/\gamma_1)\,J_1(\gamma_1 a), & g_\theta &= (\beta_2/\gamma_2)\,J_1(\gamma_2 a). \end{aligned}\right\} \tag{27}$$

The dispersion equation (26) then becomes

$$\frac{\gamma_1}{\beta_1}\frac{J_0(\gamma_1 a)}{J_1(\gamma_1 a)} - \frac{\gamma_2}{\beta_2}\frac{J_0(\gamma_2 a)}{J_1(\gamma_2 a)} = -k\left(\frac{1}{\beta_1} - \frac{1}{\beta_2}\right)\frac{K_0(ak)}{K_1(ak)}. \tag{28}$$

When $\Omega_e\tau \to \infty$ a further simplification may be made by expanding β_1 and β_2 in powers of $(\Omega_e\tau)^{-1}$. Thus, when $\nu \gg \omega$

$$\beta_1 = Ne\omega\mu_0/Bk + O(\Omega_e\tau)^{-1}, \tag{29}$$

and

$$\beta_2 = -ik\Omega_e\tau - Ne\omega\mu_0/Bk + O(\Omega_e\tau)^{-1}. \tag{30}$$

In the limit $\Omega_e\tau \to \infty$, $|\beta_2/k| \to \infty$, and $\gamma_2 \to \beta_2$; $a\beta_2 \approx iak\Omega_e\tau$ is therefore a large imaginary number provided

$$ak \gg (\Omega_e\tau)^{-1}. \tag{31}$$

When (31) is satisfied, the Bessel functions $J_0(\beta_2 a)$ and $J_1(\beta_2 a)$ in equation (28) have large imaginary arguments, and since

$$J_m(\beta_2 a)/J_n(\beta_2 a) \to 1 \quad \text{as} \quad i\beta_2 a \to \infty \text{ for all } m \text{ and } n,$$

equation (28) reduces to

$$\frac{\gamma_1}{\beta_1}\frac{J_0(\gamma_1 a)}{J_1(\gamma_1 a)} - i = -\frac{k}{\beta_1}\frac{K_0(ak)}{K_1(ak)}. \tag{32}$$

The occurrence of i in equation (32) implies that k has a non-zero imaginary part for real ω. It follows that in the regime where $\Omega_e\tau \gg 1 \gg \omega/\nu$ and $ak \gg (\Omega_e\tau)^{-1}$, wave attenuation remains finite but independent of $\Omega_e\tau$. In figure 1 an example is given showing the attenuation curve (labelled C) extrapolated to $\Omega_e\tau = \infty$ and this may be seen to be comparable to the residual damping arising from volume currents.

When $\omega/\nu \gg 1$, electron inertia determines the structure of the surface currents and the wave attenuation vanishes as $\omega/\nu \to \infty$.

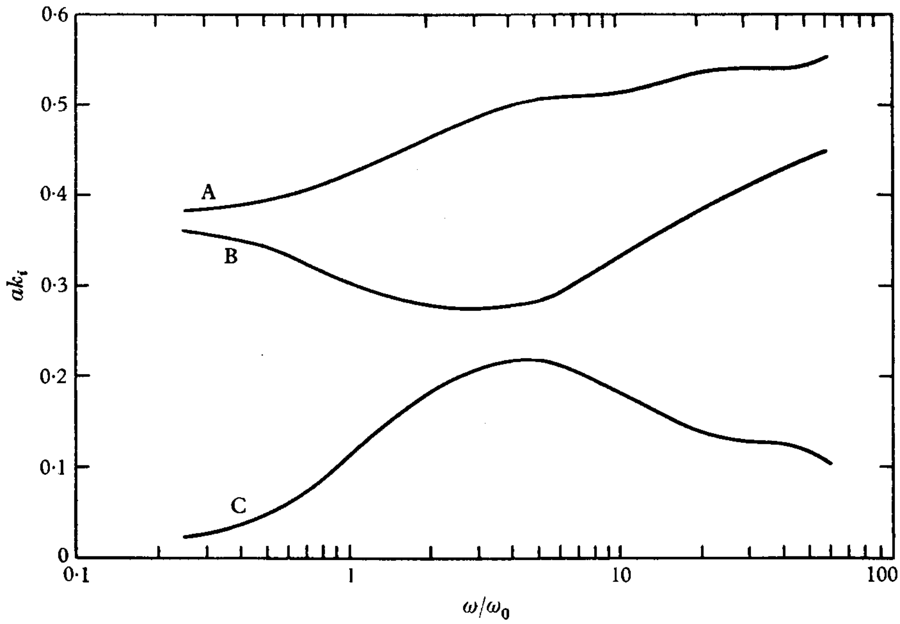

FIGURE 1. The ordinate is the attenuation constant for travelling waves, defined so that the wave amplitude decreases to $1/e$ of its initial amplitude in a distance $z = 1/k_i$. The wave frequency ω is given relative to $\omega_0 = B/ne\mu_0 a^2$. Curve A, attenuation constant for $m = 0$ and $\Omega_e\tau = 20$; curve B, attenuation constant for $m = 0$ and $\Omega_e\tau = 20$ when surface currents are ignored; curve C, attenuation constant due to surface current alone.

6. The method used for numerical solution of the dispersion relation

Numerical solutions of equation (26) were found for $m = 0$ and $m = 1$ waves of low radial wave number which satisfy the inequality $\Omega_e \gg \nu \gg \omega$. For travelling waves the real and imaginary parts of k are required for real ω, and for standing waves the real and imaginary parts of ω for real k. The computational procedure will be illustrated for the travelling $m = 0$ wave. The parameters ω, Ω_e, ν, ω_p are assumed to be specified.

If $k = k_r + ik_i$ then the dispersion equation (26), represented by $D(\omega, k) = 0$, can be expressed in the form

$$D(\omega, k) = U(\omega, k_r, k_i) + iV(\omega, k_r, k_i) = 0, \tag{33}$$

and the relation between ω, k_r and k_i must be such that U and V are simultaneously zero. By choosing a sequence of values of k_r and k_i and calculating the corresponding value of U, a polyhedral surface may be constructed. A different polyhedral surface exists for each ω. The intersections of surfaces with the $U = 0$ plane give closed curves on which $U(\omega, k_r, k_i) = 0$. A similar procedure yields the value of k_r and k_i for which $V = 0$, i.e. the curve $V(\omega, k_r, k_i) = 0$. Superposition of these two sets of closed curves gives the relation between k_r and k_i which simultaneously satisfies $U = 0$, $V = 0$. These curves intersect in pairs at right angles at two points. The intersection of these curves locate the poles and zeros of the dispersion relation, together with the branch points in a specified region of the complex plane. The two sets of curves $U = 0$, $V = 0$ were plotted and superimposed by an automatic graph plotter.

Guidance in the choice of values for k_r and k_i was obtained as follows. If the attenuation of the wave is small, solutions are expected to lie in the neighbourhood of the points for which β_1 is real.

If β_1 is assumed real, then equation (11) can be written

$$\frac{\omega\Omega_e^2}{\nu^2} k_i^2 - \frac{\Omega_e}{\nu} k_r k_i + \frac{\omega\omega_p^2}{c^2} = 0. \tag{34}$$

When $\omega \ll \nu$, (34) represents a hyperbola in the k-plane defined by

$$k_i k_r = \omega\nu\omega_p^2/\Omega_e^2 c^2. \tag{35}$$

Similarly for standing waves β_1 is real along a parabola in the ω-plane

$$\omega_i = (\nu\omega_p^2/k^2\Omega_e^2 c^2)\,\omega_r^2. \tag{36}$$

$\beta_1 = \beta_2$ is a trivial solution of the dispersion relation for all m. For this case the wave fields vanish, and the solution corresponds to a branch point in the complex k-plane.

The solutions of equation (16) for β may be regarded as functions of six independent parameters:

$$a\beta = a\beta(ak_r, ak_i, \omega_r/\Omega_e, \omega_i/\Omega_e, \nu/\Omega_e, \omega/\omega_0), \tag{37}$$

where $$\omega_0 = B/Ne\mu_0 a^2.$$

The parameter ω_r/Ω_e appears only through the electron inertia which, under the conditions studied ($\nu \gg \omega$), can be neglected. The parameter ν/Ω_e accounts for the attenuation of the waves, but only affects the dispersion curves to second order in this parameter.

Figure 2 shows the $m = 1$ dispersion function in the complex ω-plane; the first-order poles lie on a parabola and the simple zeros of the function lie near the poles. The poles were distinguished from the zeros by plotting the contour $U = \text{const.}$ (indicated by 3 in figure 2) as well as the contours $U = 0$, $V = 0$ (indicated by 1 and 2 respectively in figure 2). Since D and $D + \text{const.}$ have the same poles but different zeros, the contours 1 and 3 cut 2 in the common pole and in two other points which are the zeros of D and $D + \text{constant}$.

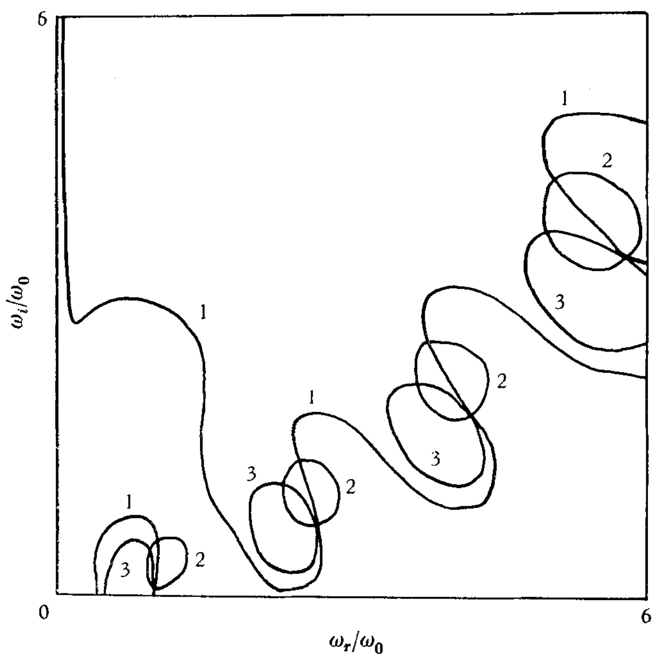

FIGURE 2. The dispersion function $D = U + iV$ (equation (33)) in the complex ω-plane for a standing wave with $m = 1$ and $\Omega_e \tau = 10$. Curve 1, contours of $U = 0$; curve 2, contours of $V = 0$; curve 3, contours of $U = 0{\cdot}1$.

Figure 3 shows the $m = 0$ dispersion function in the complex k-plane. The poles and zeros lie near a hyperbola. The point C is the branch point of the dispersion function and the line CD along which the contours of real and imaginary part run together is determined by the choice of the negative real axis as the branch cut of $\sqrt{Z}$ in the Z-plane. This method gives good starting-points for iterative solution of the dispersion relations and, if necessary, can produce arbitrarily accurate results by looking more closely at any desired root with a finer mesh of points. More accurate solutions of the dispersion relations were found by iteration using Muller's method, in which the function is approximated by a quadratic, by the rule of false position, and using a bilinear approximation to the function. The results for travelling waves are shown in figures 4–6, and for standing waves in figures 7 and 8. These methods have been incorporated in a multi-purpose programme for solving general complex eigenvalue problems (McNamara 1964).

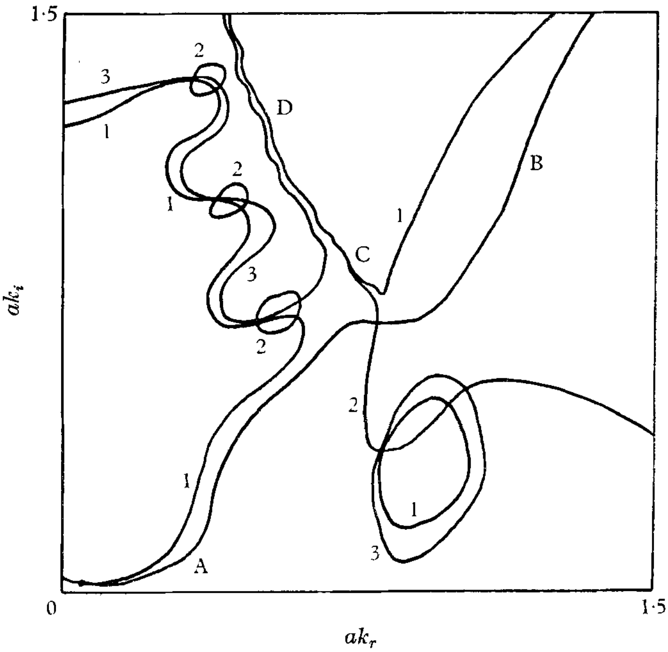

FIGURE 3. The dispersion function $D = U + iV$ (equation (33)), in the complex k-plane for a travelling wave with $m = 1$ and $\Omega_e \tau = 10$. Curve 1, contours of $U = 0$; curve 2, contours of $V = 0$; curve 3, contours of $U = 0{\cdot}1$; curve AB, contour corresponding to the trivial solution for which wave field vanishes, i.e. $\beta_1 = \beta_2$.

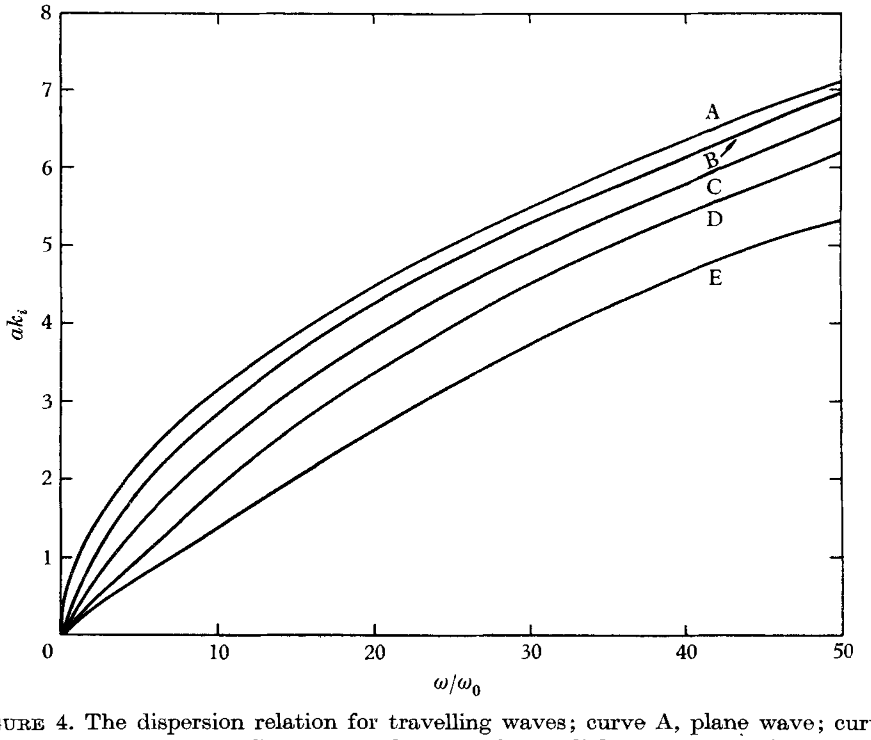

FIGURE 4. The dispersion relation for travelling waves; curve A, plane wave; curve B, $m = 0$ and $n = 1$; curve C, $m = 1$ and $n = 1$ (first radial mode); curve D, $m = 1$ and $n = 2$; curve E, $m = -1$ and $n = 2$.

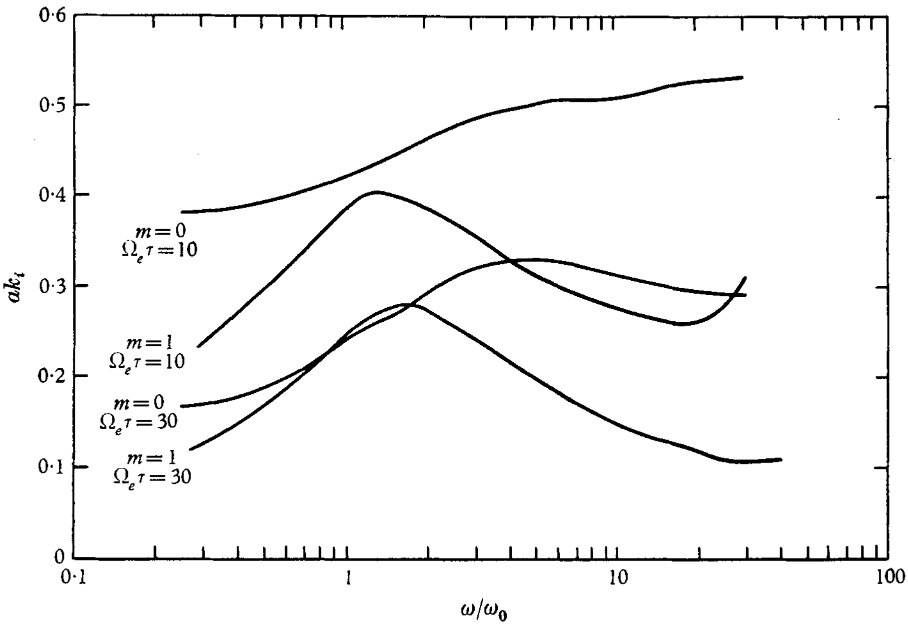

FIGURE 5. The attenuation constants for travelling waves with $m = 0$ and 1 and two values of $\Omega_e\tau$. For $\Omega_e\tau > 10$, linear interpolation in $(\Omega_e\tau)^{-1}$ may be used.

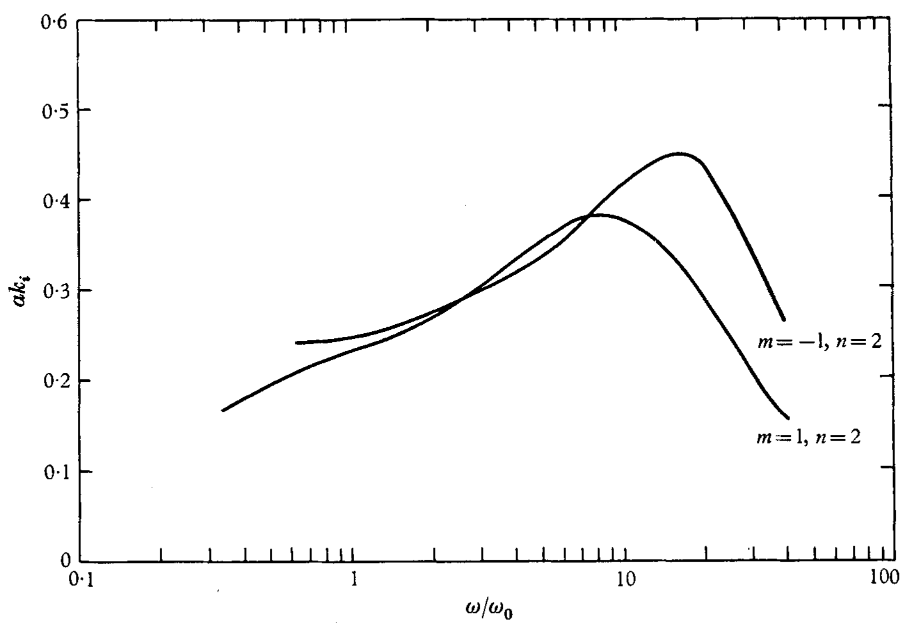

FIGURE 6. The attenuation constants for travelling waves with $m = 1$ and $m = -1$, $n = 2$ and $\Omega_e\tau = 30$.

7. The radial variation of the wave fields for $m = 0$

It was not possible, *a priori*, to determine the boundary conditions for the case of zero resistivity because the magnitude of the surface currents are not known. However, by allowing $\Omega_e \tau$ to tend to infinity the radial variation of the wave fields in the neighbourhood of the plasma vacuum interface can now be obtained.

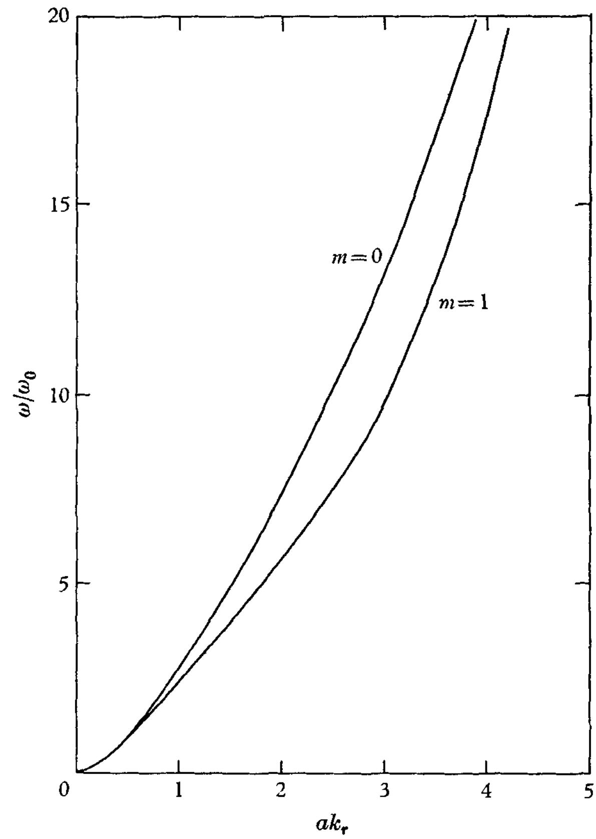

FIGURE 7. The dispersion relation for standing waves with $m = 0$, $m = 1$ and $n = 1$.

Using equations (17)–(21), (23)–(25), and assuming continuity of the wave field across the boundary ($\Omega_e \tau$ finite), the constants A_1, A_2 and A_3 may be evaluated in terms of the f and g functions at $r = a$ (equation (27)). For the case $m = 0$ the three components of the field are

$$\left.\begin{aligned} \hat{b}_r &= -\frac{ik}{\beta_1}\frac{J_1(\gamma_1 r)}{J_1(\gamma_1 a)} + \frac{ik}{\beta_2}\frac{J_1(\gamma_2 r)}{J_1(\gamma_2 a)}, \\ \hat{b}_\theta &= \frac{J_1(\gamma_1 r)}{J_1(\gamma_1 a)} - \frac{J_1(\gamma_2 r)}{J_1(\gamma_2 a)}, \\ \hat{b}_z &= \frac{\gamma_1}{\beta_1}\frac{J_0(\gamma_1 r)}{J_1(\gamma_1 a)} - \frac{\gamma_2}{\beta_2}\frac{J_0(\gamma_2 r)}{J_1(\gamma_2 a)}. \end{aligned}\right\} \tag{38}$$

An amplitude factor common to each component has been taken as unity.

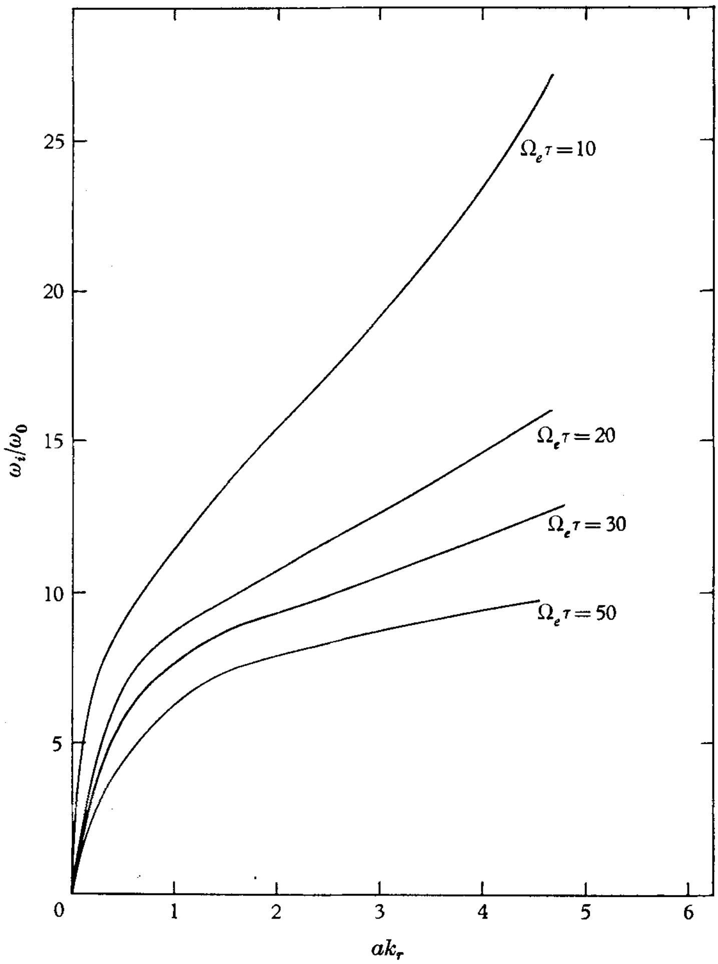

FIGURE 8. The attenuation constant for standing waves with $m = 0$, $n = 1$. Linear interpolation in $(\Omega_e\tau)^{-1}$ may be used.

In the limit $\Omega_e\tau \to \infty$ the terms containing β_2 and γ_2 simplify, since $i\beta_2 \to \infty$, $i\gamma_2 \to \infty$. For instance

$$\hat{b}_\theta \approx J_1(\gamma_1 r)/J_1(\gamma_1 a) - \exp[-ak\Omega_e\tau(1-r/a)]. \tag{39}$$

The resistive field is finite only in a boundary region of thickness

$$\delta_B \sim 1/k\Omega_e\tau = \text{wavelength}/2\pi\Omega_e\tau, \tag{40}$$

and can be written in the limit $\Omega_e\tau \to \infty$ as a discontinuous function, $d(a-r)$, where

$$\left.\begin{aligned} d(a-r) &= 1 \quad \text{if} \quad a = r, \\ &= 0 \quad \text{otherwise.} \end{aligned}\right\} \tag{41}$$

Thus as $\Omega_e\tau \to \infty$ (38) becomes

$$\left.\begin{aligned} \hat{b}_r &= -ikJ_1(\gamma_1 r)/\beta_1 J_1(\gamma_1 a), \\ \hat{b}_\theta &= J_1(\gamma_1 r)/J_1(\gamma_1 a) - d(a-r), \\ \hat{b}_z &= \gamma_1 J_0(\gamma_1 r)/\beta_1 J_1(\gamma_1 a) - i\,d(a-r). \end{aligned}\right\} \tag{42}$$

It is seen from equation (42) that both b_θ and b_z change discontinuously at the plasma boundary, the terms involving the d-functions giving the magnitude of the jump.

For finite $\Omega_e\tau$ the fields vary smoothly through the boundary and computed magnetic field profiles are shown in figure 9 for the case $\Omega_e\tau = 10$ and $m = 0$. The modulus of the fields is plotted since there is a phase difference between the two terms of the expression determining the field. The large field gradients near the boundary are apparent and correspond to large current densities near the surface.

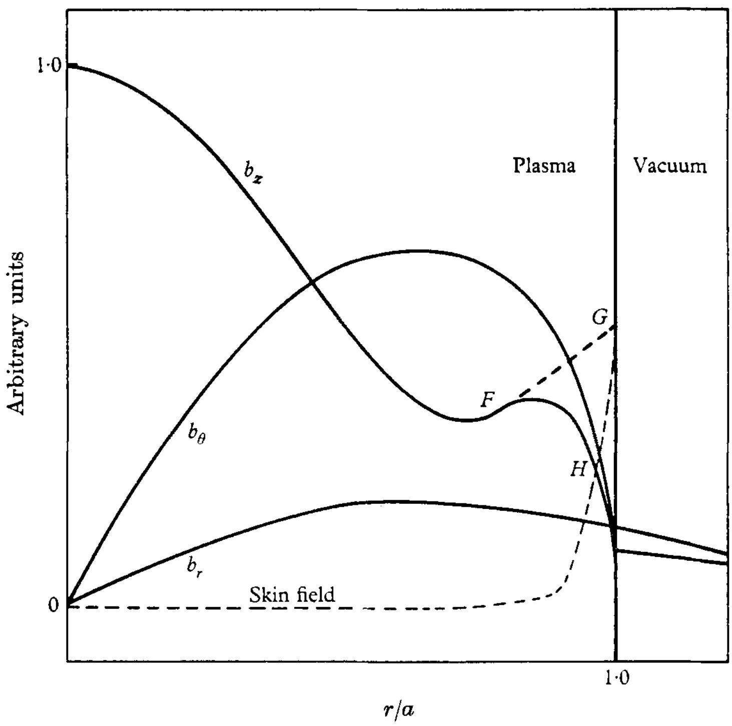

FIGURE 9. The amplitude of the magnetic field components as a function of radius for $m = 0$, $n = 1$, $\Omega_e\tau = 10$, for travelling waves. The dotted curve F to G is the amplitude of the b_z component in the absence of surface currents. The b_z field due to surface currents is referred to as the 'skin field'. The modulus of the total field is the curve from F to H.

8. Wave polarization

In an infinite uniform plasma containing a uniform magnetic field there exist two waves of opposite polarization; the right-handed one is a propagating wave and the left-handed one is evanescent in the limit $\Omega_e\tau \rightarrow \infty$. In the plasma cylinder considered here, both types of disturbance couple together to produce the resultant wave which propagates according to the dispersion relation (26). This dispersion relation is even in ak and so, for a given azimuthal mode number m, the same wave can be propagated up or down the magnetic field. However, the dispersion relation is not symmetric in the sign of the azimuthal mode number, so that negative-m modes propagate differently to positive-m modes for $m \neq 0$.

In considering the polarization of these cylindrical waves it is necessary to distinguish between the m number and the polarization. The former determines the direction of rotation of the field pattern, and the latter the direction of rotation of the magnetic field vector in the (r, θ)-plane. This is demonstrated in figure 10 where the amplitude and phase of each of the three field components of

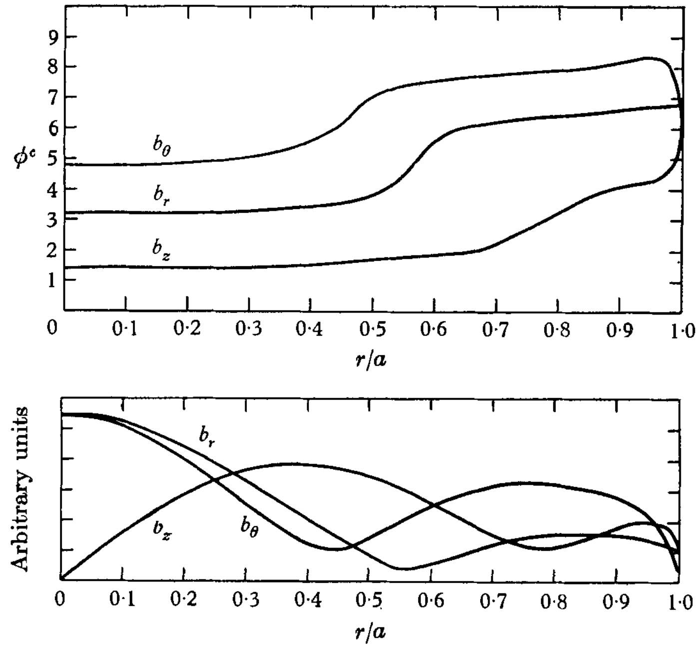

FIGURE 10. Amplitude and phase of field components for $m = 1$, $n = 2$, $\Omega_e \tau = 20$, $ak_t = 2{\cdot}56$ and $\omega/\omega_0 = 14{\cdot}5$. The top graph shows the phase ϕ of each field component as a function of radius. Note that b_θ leads b_r by 90° over most of the radius. The lower graph shows the amplitude of the field components as a function of radius.

an $m = 1$ right-handed helicon wave with one radial node are plotted as functions of radius. In the interior of the cylinder the phase of the azimuthal field is seen to lead that of the radial field representing a right polarized field at every point. The phases of the azimuthal and axial fields change rapidly near the boundary where the resistive field contributes appreciably to the total fields. For negative m numbers competition is to be expected between the right handedness of the basic propagating disturbance and the left handedness of the field pattern. This is illustrated for an $m = -1$ helicon wave in figure 11 where it is seen that the field is right polarized near the centre of the cylinder, changes through a region of left polarization to very nearly linear polarization for the rest of the interior of the cylinder, and finally becomes left circular polarized in the region of the boundary. Finally, because of the competing effects of the field polarization and direction of rotation of the field pattern, it is reasonable to find that solutions of the dispersion relation for negative m must have at least one radial node. The field patterns given in figures 9–11 and the dispersion and damping curves given below are completely representative of the infinity of possible solutions of the dispersion relation (26).

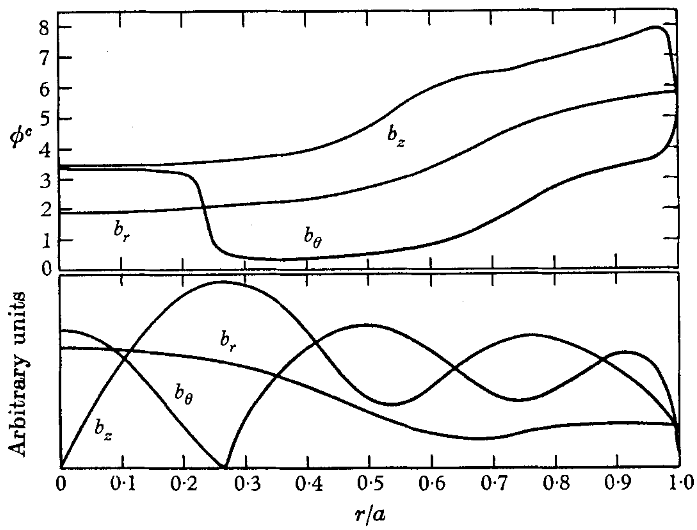

FIGURE 11. Amplitude and phase of field components for $m = -1$, $n = 2$, $\Omega_e \tau = 20$, $ak_i = 2{\cdot}0$, $\omega/\omega_0 = 14{\cdot}5$. The top graph shows the phase ϕ of each component as a function of radius. Note that b_θ leads b_r by 90° up to $r = 0{\cdot}2a$ but lags behind b_r by 180° for $r > 0{\cdot}2a$ except at the boundary. The lower graph shows the amplitude of the field components as a function of radius.

9. Discussion

The dispersion relation given above (equation (26)) was derived by including finite resistivity in the generalized Ohm's law (equation (6)). It will now be shown that if both electron inertia and plasma resistivity are omitted from Ohm's law, by setting m_e equal to zero, then the dispersion relation cannot be found.

From the linearized Ohm's law, which is

$$\mathbf{E} - (1/Ne)\,\mathbf{j} \times \mathbf{B} = 0,$$

and Maxwell's equations with displacement current neglected, an equation for $\mathbf{b}$ may be found. The previous analysis yields this equation by putting $m_e = 0$ in equation (9). For fields finite at $r = 0$, the solution for the field components involves one arbitrary constant A_1.

The magnetic field components in the vacuum are as given in equations (23)–(25) and involve another arbitrary constant A_3. With neglect of electron mass, infinite surface current densities are permissible and the boundary conditions on the θ and z components of $\mathbf{b}$ (equation 4) involve unknown surface currents j_z^* and j_θ^*. Therefore there is only one determinate boundary condition on the magnetic field, namely continuity of b_r. However, there are two constants A_1 and A_3. The remaining boundary condition (on the tangential component of the electric field) plays no part in determining the dispersion relation when displacement current is neglected, but serves only to eliminate the third constant in the vacuum electric field. Therefore the number of constants exceeds the number of boundary conditions, and the dispersion relation cannot be found.

The boundary conditions on the tangential electric field have not yet been considered because, with neglect of displacement current, they do not affect the dispersion relation. To complete the analysis of helicon wave propagation, however, this boundary condition must be introduced to determine the vacuum electric field.

Integration of the equation (4) across the boundary gives

$$[\mathbf{n} \times \mathbf{E}] = 0,$$

where $\mathbf{n}$ is the unit vector normal to the boundary. The tangential electric field is therefore continuous across the boundary, and the vacuum electric field can be determined.

Woods (1962, 1964) has introduced a perturbation electric dipole layer of density τ on the boundary in which case the boundary condition is (Stratton 1941)

$$[\mathbf{n} \times \mathbf{E}] = (1/\epsilon_0)\, \mathbf{n} \times \nabla\tau.$$

However, the electric field of the wave sets up a charge separation which may be described by a volume polarization analogous to that in a dielectric (see Appendix A for the dielectric tensor). An additional and arbitrary dipole layer is therefore superfluous to the theory and so must be discarded.

A noteworthy result is the fact that there are conditions where wave attenuation is independent of the collision frequency, provided electron inertia and ion motion can be neglected, i.e. $\nu \gg \omega$ and $\Omega_e/\nu \ll \omega/\Omega_i$.

The characteristic distance z_c for the amplitude of a wave to decrease to $1/e$ of its initial amplitude is $z_c = 1/k_i$. The curve C in figure 1 for $m = 0$ and $\Omega_e \tau \to \infty$ has a value for ak_i which does not fall below 0·1 over a range of frequencies, $1 < \omega/\omega_0 < 100$. The characteristic distance for attenuation over this range of frequencies is, therefore, never greater than $10a$. This anomalously large attenuation is due to the energy dissipated in collisions by surface currents of high current density. It would not be present if the plasma was bounded by an ideal conductor.

Appendix A

Conditions for neglect of ion motion and displacement current

The complete set of linearized equations to be solved when the ion motion and displacement current are taken into account and plasma pressure neglected are

$$\rho_0(\partial \mathbf{V}/\partial t) = \mathbf{j} \times \mathbf{B}, \tag{A 1}$$

$$\mathbf{E} = \frac{1}{Ne}\mathbf{j} \times \mathbf{B} + \eta \mathbf{j} + \frac{m_e}{Ne^2}\frac{\partial \mathbf{j}}{\partial t} - \mathbf{V} \times \mathbf{B}, \tag{A 2}$$

$$\nabla \times \mathbf{E} = -\partial \mathbf{b}/\partial t, \tag{A 3}$$

$$\nabla \times \mathbf{b} = \mu_0 \mathbf{j} + \mu_0 \epsilon_0 (\partial \mathbf{E}/\partial t), \tag{A 4}$$

$$\nabla . \mathbf{b} = 0, \tag{A 5}$$

where ρ_0 is the equilibrium ion density and $\mathbf{V}$ the plasma velocity.

Write (A 4) in the form

$$\nabla \times \mathbf{b} = \mu_0 \epsilon_0 \partial\{(\mathbf{I}+\boldsymbol{\chi}).\mathbf{E}\}/\partial t, \tag{A 6}$$

where $\boldsymbol{\chi}$ is the polarizability tensor for the magnetized plasma and $\mathbf{I}$ is the unit tensor. Consider a perturbation with a time dependence of the form $e^{i\omega t}$ so that (A 1) becomes

$$i\omega\rho_0 \mathbf{V} = \mathbf{j} \times \mathbf{B}.$$

Substituting this result into (A 2) yields

$$i\omega\epsilon_0 E = (\delta_3 + i\delta_1)\,\mathbf{j} + i\delta_2 \mathbf{j} \times \hat{\mathbf{s}}_1 - \delta_2(\hat{\mathbf{s}}_1 . \mathbf{j})\,\hat{\mathbf{s}}_1, \tag{A 7}$$

where

$$\delta_1 = \omega(\nu + i\omega)/\omega_{pe}^2, \quad \delta_2 = \omega\Omega_e/\omega_{pe}^2, \quad \delta_3 = \Omega_i^2/\omega_{pi}^2, \tag{A 8}$$

and $\hat{\mathbf{s}}_1$ is a unit vector parallel to the magnetic field. The dimensionless parameters, δ_1, δ_2 and δ_3, represent electron inertia and resistance, the Hall effect, and ion inertia respectively.

$\mathbf{j}$ may now be eliminated from (A 4) using (A 7) so that the elements of the polarization tensor are

$$\left.\begin{aligned} \chi_{11} &= \frac{1}{i\delta_1}, \quad \chi_{22} = \chi_{33} = \frac{\delta_3 + i\delta_1}{(\delta_3 + i\delta_1)^2 - \delta_2^2}, \\ \chi_{23} &= -\chi_{32} = -\frac{i\delta_2}{(\delta_3 + i\delta_1)^2 - \delta_2^2}, \quad \chi_{12} = \chi_{21} = \chi_{31} = \chi_{13} = 0. \end{aligned}\right\} \tag{A 9}$$

It is evident that ion motion may be neglected if $\delta_1 \gg \delta_3$. Written in terms of characteristic frequencies this is

$$\omega/\Omega_i \gg \Omega_e/\nu. \tag{A 10}$$

In a high temperature fully ionized gas in which a strong magnetic field is present it is not uncommon for Ω_e/ν to have values exceeding 100. In this case ion motion continues to play an important role under circumstance where the displacement current can be neglected and the wave frequency is 100 times the ion gyrofrequency.

Equations (A 1) and (A 2) are based on the assumption that the ion mass is much larger than the electron mass which is not necessarily true for an electron-hole plasma in a semi-conductor.

The condition for the neglect of displacement current is $\boldsymbol{\chi} \gg \mathbf{I}$. Since the unit tensor $\mathbf{I}$ has diagonal elements only, it follows from (A 9) that the required conditions are

$$\delta_1 \ll 1 \quad \text{and} \quad (\delta_3 + i\delta_1)/\{(\delta_3 + i\delta_1)^2 - \delta_2^2\} \gg 1. \tag{A 11}$$

For the waves studied here it is assumed that $\delta_1 \gg \delta_3$ and (A 11) reduces to

$$\delta_1 \ll 1 \quad \text{and} \quad \delta_1/(\delta_1^2 + \delta_2^2) \gg 1. \tag{A 12}$$

The following two examples illustrate circumstances for which the inequalities (A 11) are satisfied.

Typical values for the frequencies involved for helicon wave propagation in metals at liquid helium temperatures, are

$$\omega = 100, \quad \nu = 10^9, \quad \Omega_e = 2 \times 10^{11} \quad \text{and} \quad \omega_{pe} = 4 \times 10^{16},$$

so that $\delta_1 \approx 10^{-25}$ and $\delta_1/(\delta_1^2 + \delta_2^2) \approx 10^{25}$.

For helicon wave propagation in partially ionized argon gas, the following frequencies are typical

$$\omega = 10^7, \quad \nu = 10^8, \quad \Omega_e = 2\times 10^9, \quad \omega_{pe} = 6\times 10^{10},$$
$$\Omega_i = 3\times 10^4 \quad \text{and} \quad \omega_{pi} = 10^7,$$

so that $$\delta_1 = 3\times 10^{-7}, \quad (\delta_3+i\delta_1)/\{(\delta_3+i\delta_1)^2-\delta_2^2\} \approx 2\times 10^5.$$

It remains to determine the restrictions on the phase velocity of the wave $V_p = \omega/k$, for the neglect of ion motion and displacement current. The electric field can be eliminated from (A 3) and (A 6) to yield

$$\nabla\times((\mathbf{I}+\boldsymbol{\chi})^{-1}.\nabla\times\mathbf{b})-\omega^2\epsilon_0\mu_0\mathbf{b} = 0.$$

Some further manipulations yield the following equation for the z-component of $\mathbf{b}$:

$$\left\{i\delta_1\left(\delta_3+i\delta_1+\frac{\delta_2^2}{1+\delta_3+i\delta_1}\right)D^2+\left[\epsilon_0\mu_0\omega^2\{1-i\delta_1(\delta_3+i\delta_1)\}+k^2\frac{\{\delta_2^2-\delta_3(\delta_3+i\delta_1)\}}{1+\delta_3+i\delta_1}\right]\right.$$
$$\left.\times D-\epsilon_0\mu_0\omega^2\left(i\delta_1\mu_0\epsilon_0\omega^2+\frac{k^2\delta_3}{1+\delta_3+i\delta_1}\right)\right\}b_z = 0, \quad \text{(A 13)}$$

where $D = \nabla^2+\epsilon_0\mu_0\omega^2$.

Inspection of equation (A 13) yields two further well known conditions on the phase velocity of the waves. For ion motion to be negligible, the phase velocity must be much greater than the Alfvén speed, that is

$$V_p^2 \gg B^2/\mu_0\rho_0. \quad \text{(A 14)}$$

For displacement current to be negligible the phase velocity must be much less than the velocity of light *in vacuo*

$$V_p \ll (\mu_0\epsilon_0)^{-\frac{1}{2}}.$$

Finally, it can be seen that if either the displacement current or ion motion terms are small compared with the resistive terms so that

$$\delta_1 > \delta_3+\delta_2^2/(1+\delta_3), \quad \text{(A 15)}$$

then the highest derivatives in (A 13) will be of the form $\nu^2\nabla^4$ instead of $\nu\nabla^4$ and there will be an appreciable contribution to wave attenuation from surface currents. This was the case discussed in § 5 where, in addition, $\delta_1 \ll \delta_2$, i.e. $\Omega_e/\nu \gg 1$.

Appendix B

Conditions for neglect of surface current in a magnetically confined plasma

We consider a cylindrical plasma with a small electron pressure gradient and carrying a small steady confining current $\mathbf{J}_0$ in addition to the wave current $\mathbf{j}$. Ohm's law is then as follows:

$$\mathbf{E} = (1/Ne)[\mathbf{J}_0\times\mathbf{b}+\mathbf{j}\times\mathbf{B}_0]+\eta\mathbf{j}-(1/Ne)\nabla P_e, \quad \text{(B 1)}$$

where the usual linearization has been effected.

If δ_c = thickness of current layer confining the plasma (in practice J_0 would be mainly a surface current when $\Omega_e \tau \gg 1$), $\delta_B = 1/k\Omega_e \tau$ = boundary-layer thickness of helicon wave when $\Omega_e \tau \gg 1$, then the most stringent conditions on J_0 will clearly obtain when $\delta_c > \delta_B$. Comparison of the terms in $\mathbf{J}_0$ with those in $\mathbf{j}$ in equation (1) gives:

$$\frac{|(1/Ne)\,\mathbf{J}_0 \times \mathbf{b}|}{|\eta \mathbf{j}|} \sim \Omega_e \tau (\mu_0 J_0/kB_0) \quad \text{for} \quad r < |a-\delta_B|,$$

$$\sim \mu_0 J_0/kB_0 \quad \text{for} \quad a > r > |a-\delta_B|,$$

$$\frac{|(1/Ne)\,\mathbf{J}_0 \times \mathbf{b}|}{|(1/Ne)\,\mathbf{j} \times \mathbf{B}_0|} \sim \mu_0 J_0/kB_0 \quad \text{for} \quad r < |a-\delta_B|,$$

$$\sim (1/\Omega_e \tau)\, \mu_0 J_0/kB_0 \quad \text{for} \quad a > r > |a-\delta_B|.$$

We have used that ratio b/j found in the above theory for $J_0 = 0$. The terms in J_0 are then negligible compared to those in j provided

$$\mu_0 J_0/kB_0 \ll (\Omega_e \tau)^{-1}.$$

This is the restriction to be placed on J_0 if it is not to sensibly affect helicon propagation.

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